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LETTER TO THE EDITOR

Magnetic properties of the one-dimensional supersymmetric *t–J* model

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Abstract. Starting with the known exact solution of the one-dimensional supersymmetric t–J model, we investigate ground-state properties of the system in the presence of an external magnetic field. By analytical calculations we find a representation of the magnetization curves and the susceptibility using the dressed charge matrix and the charge and spin densities; the dressed properties are given by a set of coupled integral equations which are derived from the Bethe ansatz equations in the thermodynamic limit. Some special results are found analytically, namely the magnetic field at which saturation occurs and the corresponding susceptibility. Furthermore we show the vanishing of the lower critical field. The dependence of magnetization and susceptibility on the magnetic field is calculated numerically for various particle densities.

According to Anderson [1], the t–J model [2, 3] is supposed to incorporate in a very simple form the essential elements responsible for high T_c superconductivity. The model consists of a kinetic term describing next-neighbour hopping of electrons with transfer energy t and a potential term describing a spin exchange interaction of strength J also between next-neighbour sites.

In the following, we present the results of our investigations of the model in one dimension in an external magnetic field. As its supersymmetric points $2t=\pm J$, the model is exactly solvable using the Bethe ansatz technique. We examine the case $2t=J$ which represents a strong antiferromagnetic exchange interaction.

Consider a linear chain of L sites which is occupied by $N \leqslant L$ electrons, such that $N_\uparrow$ ($N_\downarrow$) is the number of up (down) spins. Double occupation of the sites is excluded, thus simulating an infinite on-site Coulomb repulsion, and periodic boundary conditions are imposed. The Hamiltonian of the system in an external magnetic field H is

$$\mathscr{H} = \mathscr{P} \sum_{j=1}^{L} \left\{ -t \sum_{\sigma=\uparrow,\downarrow} (c_{j+1,\sigma}^{+} c_{j\sigma} + c_{j\sigma}^{+} c_{j+1,\sigma}) + J(\boldsymbol{s}_j \cdot \boldsymbol{s}_{j+1} - \tfrac{1}{4} n_j n_{j+1}) - \tfrac{1}{2} H (n_{j\uparrow} - n_{j\downarrow}) \right\} \mathscr{P} \qquad (1)$$

where the operators $c_{i\sigma}^{+}$ and $c_{i\sigma}$ create and annihilate an electron on site i with spin σ. n_i is the particle number (which equals zero or one) and $\boldsymbol{s}_i$ the spin at site i. The projector $\mathscr{P}$ ensures the exclusion of doubly occupied sites.

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For $J=2t$ the model is solved by Bethe ansatz [2, 3, 4]. The energy eigenstates are given by a Bethe ansatz wavefunction with N different wavenumbers k_i which in the following are parametrized by the moments $v_i=\frac{1}{2}\cot(k_i/2)$. In the thermodynamic limit, with $N\to\infty$ while $n_\downarrow=N_\downarrow/L=$ constant, the ground state *without* magnetic field is characterized by a set of $N/2$ pairs of complex wavenumbers $k_\alpha^\pm$ which describe singlet pairs in k-space. The corresponding moments $v_\alpha^\pm$ form so-called *2-strings* [4, 5],

$$v_\alpha^\pm=\lambda_\alpha\pm\mathrm{i}/2 \tag{2}$$

where the $N/2$ spin rapidities λ_α are real numbers. The zero-field ground state has no magnetization. Applying a magnetic field H to the system creates a number of $N-2N_\downarrow$ unpaired spin-up electrons which leads to a non-vanishing total magnetization $S=(N_\uparrow-N_\downarrow)/2$ of the system. In contrast to the complex moments of the singlet pairs, the moments v_j of the unpaired spin-up electrons are real numbers. The two sets of moments are determined from the Bethe ansatz equations [4]

$$\left(\frac{v_j+\mathrm{i}/2}{v_j-\mathrm{i}/2}\right)^L=\prod_{\beta=1}^{N_\downarrow}\frac{v_j-\lambda_\beta+\mathrm{i}/2}{v_j-\lambda_\beta-\mathrm{i}/2} \tag{3}$$

$$\left(\frac{\lambda_\alpha+\mathrm{i}}{\lambda_\alpha-\mathrm{i}}\right)^L=-\prod_{j=1}^{N-2N_\downarrow}\frac{\lambda_\alpha-v_j+\mathrm{i}/2}{\lambda_\alpha-v_j-\mathrm{i}/2}\prod_{\beta=1}^{N_\downarrow}\frac{\lambda_\alpha-\lambda_\beta+\mathrm{i}}{\lambda_\alpha-\lambda_\beta-\mathrm{i}} \tag{4}$$

with $j=1,\dots,N-2N_\downarrow$ and $\alpha=1,\dots,N_\downarrow$. The total energy of the system, $E=-2t\sum_{j=1}^{N}\cos k_j-HS$, can be expressed by

$$\frac{E}{2t}=-\left(N-\sum_{j=1}^{N-2N_\downarrow}\frac{2}{4v_j^2+1}-\sum_{\alpha=1}^{N_\downarrow}\frac{1}{\lambda_\alpha^2+1}\right)-\frac{h}{2}(N-2N_\downarrow) \tag{5}$$

where h is given in units of $2t$, i.e. $H=2th$. We notice that the field h influences the energy in two ways. One contribution is the Zeeman energy proportional to the field, but the field also changes the distribution of the moments. In the thermodynamic limit, equations (3), (4) become a set of two coupled Fredholm integral equations for the continuous distribution functions of the moments v and λ which lie densely on the real axis:

$$\rho_1(v)=\frac{1}{2\pi}\frac{1}{v^2+1/4}-\frac{1}{2\pi}\int_{[B]}\frac{1}{(v-\lambda')^2+1/4}\rho_2(\lambda')\,\mathrm{d}\lambda' \tag{6}$$

$$\rho_2(\lambda)=\frac{1}{\pi}\frac{1}{\lambda^2+1}-\frac{1}{2\pi}\int_{[Q]}\frac{1}{(\lambda-v')^2+1/4}\rho_1(v')\,\mathrm{d}v'-\frac{1}{\pi}\int_{[B]}\frac{1}{(\lambda-\lambda')^2+1}\rho_2(\lambda')\,\mathrm{d}\lambda' \tag{7}$$

with the shorthand notation $[Q]=(-\infty,-Q)\cup(Q,\infty)$, and $[B]$ correspondingly. The two integral boundaries Q and B parametrize the electron and magnetization densities $n=N/L$ and $s=S/L$ via the normalization equations:

$$\int_{[Q]}\rho_1(v)\,\mathrm{d}v=n-2n_\downarrow=2s \tag{8}$$

$$\int_{[B]}\rho_2(\lambda)\,\mathrm{d}\lambda=n_\downarrow=\frac{n}{2}-s. \tag{9}$$

The above equations define implicit functions $n(Q,B)$ and $s(Q,B)$ with Q and B being in the interval $(0,\infty)$.

We now derive equations† describing the magnetic field $h(Q, B)$ and the susceptibility $\chi(Q, B)$ which, together with (8), (9), yield an implicit representation of the magnetization curves $s(h; n)$ and the susceptibility $\chi(h; n)$, parametrized by Q and B. The procedure has been developed and applied to the attractive Hubbard model in [8] which work we shall follow closely in the following.

The structure of equations (6), (7) can be emphasized by writing them in vector notation with

$$\boldsymbol{\rho}(v, \lambda) = \begin{pmatrix} \rho_1(v) \\ \rho_2(\lambda) \end{pmatrix}$$
$$\boldsymbol{\rho}(v, \lambda) = \boldsymbol{\rho}_0(v, \lambda) + \mathbf{K}(v, \lambda \mid v', \lambda') \otimes \boldsymbol{\rho}(v', \lambda') \tag{10}$$

where $\mathbf{K}$ denotes the 2×2-matrix consisting of the integral kernels and the product $\otimes$ stands for matrix multiplication and integration of the variables v' and λ' within the ranges $[Q]$ and $[B]$, respectively. The vector $\boldsymbol{\rho}_0(v, \lambda)$ denotes the inhomogeneous parts of the integral equations. Analogous equations hold for the dressed energy $\boldsymbol{\varepsilon}(v, \lambda)$ and the dressed charge matrix $\Xi(v, \lambda)$. These quantities are defined by the appropriate inhomogeneous terms, i.e. the bare energy

$$\boldsymbol{\varepsilon}_0 = \begin{pmatrix} \dfrac{1/2}{v^2 + 1/4} - \dfrac{h}{2} - 1 \\ \dfrac{1}{\lambda^2 + 1} - 2 \end{pmatrix} \tag{11}$$

and the bare charge matrix Ξ_0 is just the 2×2 unit matrix. Denoting the columns of Ξ by $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ and introducing the vector $\boldsymbol{\zeta} = \boldsymbol{\xi}^{(1)} + 2\boldsymbol{\xi}^{(2)}$, the energy density $e = E/2tL$ (cf (5)) as well as n and s (equations (8), (9)) can be written as

$$e = \boldsymbol{\rho}_0^{\mathrm{T}} \otimes \boldsymbol{\varepsilon} \tag{12}$$

$$n = \boldsymbol{\rho}_0^{\mathrm{T}} \otimes \boldsymbol{\zeta} \tag{13}$$

$$s = \frac{1}{2} \boldsymbol{\rho}_0^{\mathrm{T}} \otimes \boldsymbol{\xi}^{(1)}. \tag{14}$$

In order to find an expression $h(Q, B)$ for the field, one has to minimize the energy with respect to the magnetization (h and n being fixed):

$$\left. \frac{\partial E(n, s)}{\partial s} \right|_{n = \text{constant}} = 0. \tag{15}$$

This differentiation can be performed via differentiating equation (12) with respect to Q and B. It should be pointed out that any component of the dressed quantities is a function of Q *and* B because it is defined by a system of coupled integral equations containing Q and B as limits of integration. Differentiation finally leads to the equation

$$h(Q, B) = \pi \frac{\rho_1(Q)\zeta_2(B) - \rho_2(B)\zeta_1(Q)}{\det \Xi(Q, B)} \tag{16}$$

taking into account the relation

$$\boldsymbol{\varepsilon} = -\boldsymbol{\zeta} - \frac{h}{2} \boldsymbol{\xi}^{(1)} + \pi \boldsymbol{\rho} \tag{17}$$

† A more detailed description can be found in [6] and will be published elsewhere [7].

which holds for the t-J model. Equation (16) specifies the magnetic field h which causes the magnetization s. As one notices, for explicit calculations the dressed values must be evaluated at the pseudo Fermi surfaces, i.e. at the points $v=Q$ and $\lambda=B$. The magnetic susceptibility $\chi=(\partial s/\partial h)|_n$ can be derived similarly as

$$\chi^{-1}(Q,B)=-\frac{\pi}{2[\det \Xi(Q,B)]^2}\left(\frac{\bar{\rho}_1(Q)[\zeta_2(B)]^2}{\rho_1(Q)}+\frac{\bar{\rho}_2(B)[\zeta_1(Q)]^2}{\rho_2(B)}\right). \quad (18)$$

The vector $\bar{\boldsymbol{\rho}}$ denotes the 'dressed derivative' of $\boldsymbol{\rho}$, i.e. it is determined by integral equations corresponding to (10) where the inhomogeneous terms are replaced by the derivatives of $\boldsymbol{\rho}_0$ with respect to v and λ. We mention that the chemical potential $\mu=(\partial e/\partial n)|_h$ can be calculated in the same way resulting (in units of $2t$) in

$$\mu(Q,B)=-1-\pi\frac{\rho_1(Q)\xi_2^{(1)}(B)-\rho_2(B)\xi_1^{(1)}(Q)}{2\det \Xi(Q,B)}. \quad (19)$$

For general values of Q and B, the dressed properties must be calculated numerically; to perform this work we used the equivalent Sutherland representation [3, 5] of the Bethe ansatz equations which is favourable for the numerical treatment. In the following, we deal with special choices of Q or B which can be treated analytically.

The case of *magnetic saturation* corresponds to $B=\infty$. All spins are pointing into the direction of the external field so that the magnetization yields $s=n/2$. The integral equations of type (6), (7) can be solved in closed form as in this case the first components of the dressed quantities are identical to the bare quantities. The particle density n turns out to be related to the parameter Q by $n(Q)=(2/\pi)\cot^{-1}(2Q)$. Finally one can determine the field h_s at which saturation occurs:

$$h_s(n)=2\sin^2\left(\frac{n\pi}{2}\right). \quad (20)$$

This is a new exact result which is identical to the limit $U\to 0$ of the corresponding field of the Hubbard model [8-10]. It is remarkable that the saturation field h_s is the same as for free electrons, i.e. for $J=0$ and no projection $\mathcal{P}$ in (1). The susceptibility and chemical potential at saturation can also be calculated. They are given by

$$\chi_s(n)=\frac{1}{\pi\sin(n\pi)} \quad (21)$$

$$\mu_s(n)=-\cos^2\left(\frac{n\pi}{2}\right) \quad (22)$$

which are again the free-electron values.

The case $Q=\infty$ corresponds to the *onset of magnetization*, $s=0$, and the connected integral equations can be solved using the Wiener-Hopf method. By that, the vanishing of the lower critical field h_c can be proved for arbitrary density n (which now depends only on B). It can be shown that $h(Q,B)\propto e^{-\pi Q}$ as $Q\to\infty$.

Figures 1 and 2 give a graphical representation of s/n and χ/n plotted against h for various fixed densities. All these numerically determined curves end at the analytically found saturation points which in the plots are marked by dots. In the case $n=1$ we recover the results for the antiferromagnetic Heisenberg model [10].

As can be recognized, the magnetization values of the t-J model for fields close to zero or close to the saturation field hardly differ from the values obtained for free fermions. For small densities, the model furnishes only a small correction to free

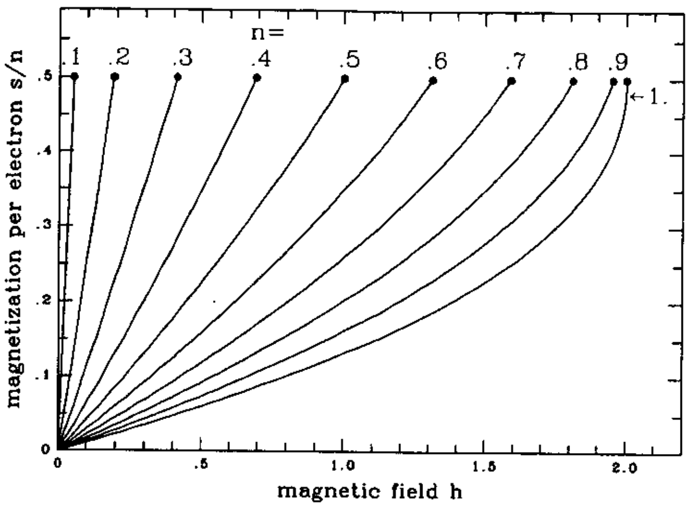

Figure 1. Magnetization curves of the t–J model for various densities n.

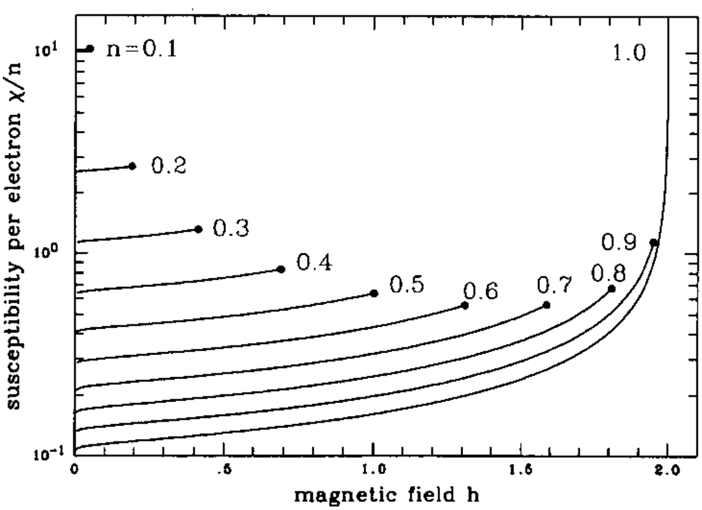

Figure 2. The susceptibility of the t–J model for various densities n.

fermions even in the whole range of $h \in (0, h_s)$. For $n \leqslant 0.4$, the difference is less than 1% for all fields despite the large exchange energy. One may speculate that this behaviour is related to charge–spin separation [11]. We suppose the free fermion behaviour to vanish as soon as elementary excitations from the ground state are considered.

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